Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Taylor series

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} -$$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots$$

$$e^{i\theta} = 1 + i\theta + -\frac{\theta^2}{2!} + -\frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} + -\frac{i\theta^7}{7!} + \frac{\theta^8}{8!} + \cdots$$

$i^2 = -1$ $\quad i^5 = i$

$i^3 = -i$ $\quad i^6 = -1$

$i^4 = 1$ $\quad i^7 = -i$

$i^8 = 1$

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)$$

$$= \cos\theta + i\sin\theta$$ Euler form.

$$g_n = \frac{F_n}{F_{n+2}} \qquad g_n = \_\_$$

$$F_{n+2} = F_{n+1} + F_n$$ recursive formula.

Easier: $$B_n = \frac{F_n}{F_{n+1}}$$

Question: Solve $a_{n+1} = 4a_n - 4a_{n-1}$

$a_0 = 2$, $a_1 = 3$

Try $a_n = r^n \Rightarrow r^{n+1} = 4r^n - 4r^{n-1}$

$$r^{n+1} - 4r^n + 4r^{n-1} = 0$$

divide by r^{n-1}

$$r^2 - 4r + 4 = 0$$

$$(r-2)(r-2) \qquad r_1 = 2,\ r_2 = 2$$

General Solution: $\cancel{a_n = c_1 2^n + c_2 2^n}$ does not work

$$a_n = c_1 2^n + c_2 n 2^n$$

Put initial conditions in:

$$a_0 = 2 = c_1 2^0 + c_2 0 \cdot 2^0 \overset{= c_1}{\Rightarrow} c_1 = 2$$

$$a_1 = 3 = c_1 2^1 + c_2 \cdot 1 \cdot 2^1 = 2c_1 + 2c_2$$

$$3 = 4 + 2c_2$$
$$-1 = 2c_2$$
$$\boxed{c_2 = -\tfrac{1}{2}}.$$

$\Rightarrow$ Solution: $a_n = 2\cdot 2^n - \frac{1}{2}\cdot n\cdot 2^n.$

Check: $a_0 = 2\cdot 1 - \frac{1}{2}\cdot 0\cdot 2^0 = 2. \checkmark$

$$a_1 = 2\cdot 2^1 - \tfrac{1}{2}\cdot 1\cdot 2^1 = 4 - 1 = 3 \checkmark$$
$$a_2 = 2\cdot 2^2 - \tfrac{1}{2}\cdot 2\cdot 2^2 = 8 - \tfrac{1}{2}8 = 8-4 = 4$$
$$a_3 = 2\cdot 2^3 - \tfrac{1}{2}\cdot 3\cdot 2^3 = 16 - 12 = 4 \checkmark$$

$$a_{n+1} = 4a_n - 4a_{n-1}$$

$n=2$ $\quad a_3 = 4a_2 - 4a_1 = 4\cdot 3 - 4\cdot 2 = \boxed{4}. \checkmark$

Inhomogenous Linear Recursion Formulas.

Looks like

$$a_n = \underbrace{c_0 a_{n-1} + c_1 a_{n-2} + \cdots + c_k a_{n-k}}_{\text{homogeneous part}} + F(n)$$

The associated homogeneous equation is

$$a_n = c_0 a_{n-1} + c_1 a_{n-2} + \cdots + c_{k-1} a_{n-k}$$

How do we solve this?

1. Solve associated homogeneous recursion for the general solution.
   (Try $a_n = r^n$, get equation for r, solve for the roots....)

homogeneous solution. $a_n^h = k_1 \_ + k_2 \_ \cdots + k_n \_$

(2) Find @ particular solution b_n

so that $b_n = c_0 b_{n-1} + c_1 b_{n-2} \cdots c_{k-1} b_{n-k} + F(n)$

(3) Then the general solution of the inhomogeneous recursion is

$$a_n = a_n^h + b_n$$

homogeneous particular

Why does this work?

$$a_n^h + b_n = c_0 a_{n-1}^h + c_1 a_{n-2}^h + \cdots + c_{k-1} a_{n-k}^h$$
$$+ c_0 b_{n-1} + c_1 b_{n-2} + \cdots + c_{k-1} b_{n-k} + F(n)$$

$$= c_0(a_{n-1}^h + b_{n-1}) + c_1(a_{n-2}^h + b_{n-2}) +$$
$$\cdots + c_{k-1}(a_{n-k}^h + b_{n-k}) + F(n).$$

Example. Solve $a_n = a_{n-1} + 2a_{n-2} + 3^n$. $a_0 = -1$, $a_1 = 3$

(1) homogeneous $a_n = a_{n-1} + 2a_{n-2}$

try $a_n = r^n \Rightarrow$

$$r^n = r^{n-1} + 2r^{n-2}$$

$$r^n - r^{n-1} - 2r^{n-2} = 0$$

$$r^2 - r - 2 = 0 \Rightarrow (r-2)(r+1) = 0$$

$r_1 = 2,\ r_2 = -1$

homogeneous solution: $C_1 2^n + C_2(-1)^n$

(2) Need particular solution.

Try $a_n = K \cdot 3^n$

$$K \cdot 3^n = K \cdot 3^{n-1} + 2 \cdot K \cdot 3^{n-2} + 3^n$$

$$3^n \cdot K = 3^n \cdot K \cdot 3^{-1} + 3^n \cdot 2K \cdot 3^{-2} + 3^n \cdot 1$$

$$\Rightarrow K = \frac{K}{3} + \frac{2K}{9} + 1$$

$$K - \frac{K}{3} - \frac{2K}{9} = 1$$

$$K\left(1 - \frac{1}{3} - \frac{2}{9}\right) = 1$$

$$\frac{9}{9} - \frac{3}{9} - \frac{2}{9} \Rightarrow K \cdot \frac{4}{9} = 1$$

$$K = \frac{9}{4}.$$

Particular solution is $\boxed{a_n = \frac{9}{4} \cdot 3^n}$

(3) General solution = homogeneous + particular

$$\boxed{a_n = C_1 2^n + C_2(-1)^n + \frac{9}{4} 3^n}$$

Initial Condition: $a_0 = -1$, $a_1 = 3$

$n=0$ $\quad -1 = C_1 \cdot 2^0 + C_2(-1)^0 + \frac{9}{4} \cdot 3^0$

$$-1 = C_1 + C_2 + \frac{9}{4}$$

$$-\frac{13}{4} = C_1 + C_2$$

$n=1 \quad 3 = c_1 \cdot 2^1 + c_2(-1)^1 + \frac{9}{4} \cdot 3^1$

$3 = 2c_1 - c_2 + \frac{27}{4}$

$-\frac{15}{4} = 2c_1 - c_2$

$c_1 + c_2 = -\frac{13}{4}$

$2c_1 - c_2 = -\frac{15}{4}$

add $\quad 3c_1 = -\frac{28}{4} = -\frac{14}{2} = -7$

$c_1 = -\frac{7}{3}$

$-\frac{7}{3} + c_2 = -\frac{13}{4} \Rightarrow c_2 = -\frac{13}{4} + \frac{7}{3}$

$c_2 = \frac{-39+28}{12}$

$c_2 = -\frac{11}{12}$.

Solution: $\boxed{a_n = -\frac{7}{3} \cdot 2^n + -\frac{11}{12}(-1)^n + \frac{9}{4} \cdot 3^n}$
